Fibrational Framework for Nested Alternating Fixed Points and (Bi)Simulation Notions for Büchi Automata

M1 internship report

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Introduction

Background. Bisimulations [1, 2] are a widely-used tool in software verification, as they allow to show that two transition systems, for example Kripke frames (sets of states equipped with transition functions between states) or automata (Kripke frames where some states are chosen to be accepting), have the same behavior (whether they accept the same input words for instance). In other words, by modeling a program as a transition system, one may show using bisimulations that a program behaves as expected [3].

In practice, Kripke frames or classic automata are usually too simple a framework to fully express the semantics of a program. Extensive work has thus been done in order to generalize the notion of bisimulation, mainly in three different directions : *(i)* to new kinds of transition systems, for example probabilistic ones such as Markov chains [4] ; *(ii)* to new kinds of expressivity, for example bisimulation distances [5] that are not only qualitative (they express whether two states have the exact same behavior or not), but also quantitative (they express how much these states differ in behavior) ; and *(iii)* to new kinds of accepting conditions : for instance, for Büchi automata (an automata where an infinite word is accepted if a corresponding run goes an infinite amount of times through accepting states), modeling reactive systems, two states related by the fair or the delayed bisimulations will accept the same words [6].

In many cases finding under-approximations of the bisimulation is enough, as, for instance, to show that two states have the same behavior it suffices to compute whether these two states in particular are bisimilar but it is not necessary to compute the whole relation. For several notions of bisimulation, these under-approximations can be characterized through game theory by winning positions in well-crafted games [7, 8, 9, 10].

Generalizing this, members of the Erato MMSD Project introduced a category theoretical framework, codensity liftings, that instantiates numerous notions of bisimulation [11] (with various kinds of transition systems *(i)* and expressivity *(ii)*), allowing for their generalization through categorical reparameterization. Using this framework, a categorical link between bisimulations and game theory was then established by introducing the codensity safety game that characterizes these bisimulation notions [12]. However, these works only deal with bisimulations that can be expressed as greatest fixed points, which is not enough for complex accepting conditions *(iii)*. For example, the fair and delayed bisimulations fail to be instantiated by this framework as they are characterized by parity games, i.e. nested alternating fixed points [13].

Goals. During my internship, under the supervision of Ichiro Hasuo and Jérémy Dubut, I thus visited the Tōkyō site of the Erato MMSD Project in order to extend these previous works to nested alternating fixed points by characterizing their under-approximants as winning positions in well-crafted games. The hope was then that this framework would instantiate the fair and delayed bisimulations and thus enable their generalizations, in particular to bisimulation distances for Büchi-like probabilistic transition systems.

Contents

1 Preliminaries

The first part of the internship consisted in a bibliographical work whose aim was to get a better understanding of tools and results that could later be used to achieve our goals. These tools and results are presented in this section.

It is assumed that the reader is already familiar with basic notions of order theory, category theory and modal logic.

1.1 Coalgebraic Modal Logic

The coalgebraic μ-calculus $\mathbf{C}\mu_{\Gamma,\Lambda}$ [13] is an extension of classic modal logic to the categorical setting.

1.1.1 Coalgebras [14]

Let $F : \mathbb{B} \to \mathbb{B}$ be an endofunctor over a category $\mathbb{B}$.

Definition 1.1.1 (F-coalgebra)**.** *An F-coalgebra over an object X of $\mathbb{B}$ is an arrow $c : X \to FX$.*

Intuitively, X is a state space, F a behavior type and c a transition structure. An F-coalgbera is thus a transition system of the behavior type F.

Example 1.1.2. *We use π and π' for the projections along the first and second components of a binary product.*

A (non-labelled) Büchi automaton is a tuple (X, c) where X is the set of states and $c : X \to 2 \times \mathcal{P}X$ (a coalgebra for the product of 2 and the powerset functor $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$) is the transition function : $(\pi' \circ c)(x)$ is the subset of states reachable from x and $(\pi \circ c)^{-1}(1)$ is the set of accepting states.

Similarly, a (non-labelled) Markov chain is a tuple (X, c) where X is the set of states and $c : X \to 2 \times \mathcal{D}_{\leq 1}X$ (a coalgebra for the product of 2 and the subdistribution functor $\mathcal{D}_{\leq 1} : \mathbf{Set} \to \mathbf{Set}$) is the transition function : $(\pi' \circ c)(x)(y)$ is the probability of reaching y from x and $(\pi \circ c)^{-1}(1)$ is the set of accepting states.

1.1.2 Coalgebraic μ-Calculus $\mathbf{C}\mu_{\Gamma,\Lambda}$

Let $F : \mathbf{Set} \to \mathbf{Set}$ be an endofunctor and let Ω be a complete lattice.

Definition 1.1.3 (Λ, Γ)**.** *A* modal signature Λ *over F is a ranked alphabet $\Lambda = (\Lambda_n)_{n\in\omega}$. An element $\lambda \in \Lambda_n$ is an n-ary* modality *and $|\lambda|$ is its arity n.*

We assume that a modal signature comes with its interpretation*. Assigned to each $\lambda \in \Lambda$ is a natural transformation*

$$[\![\lambda]\!] : (\Omega^-)^{|\lambda|} \to \Omega^{F-}$$

whose components are monotone with respect to (pointwise extensions of) the order $\sqsubseteq$ of the domain Ω of truth values.

Similarly, a propositional signature *is a ranked alphabet Γ where each $\gamma \in \Gamma$ is called a* propositional connective*. Unlike a modal signature, each $\gamma \in \Gamma$ is interpreted by a function $[\![\gamma]\!] : \Omega^{|\gamma|} \to \Omega$; we require that $[\![\gamma]\!]$ be monotone.*

Definition 1.1.4 ($\mathbf{C}\mu_{\Gamma,\Lambda}$)**.** *The language of $\mathbf{C}\mu_{\Gamma,\Lambda}$ over the modal signature Λ and the propositional signature Γ is given by the following set of formulæ.*

$$\varphi, \varphi_i := u \mid \boxdot_\gamma \left(\varphi_1, \ldots, \varphi_{|\gamma|}\right) \mid \heartsuit_\lambda \left(\varphi_1, \ldots, \varphi_{|\lambda|}\right) \\ \mid \mu u.\, \varphi \mid \nu u.\, \varphi$$

Here u is a (fixed point) variable. The notations $\boxdot_\gamma$ (for $\gamma \in \Gamma$) and $\heartsuit_\lambda$ (for $\lambda \in \Lambda$) are to distinguish propositional connectives (the former) from modalities (the latter).

Example 1.1.5. *Take $F = \mathcal{P}(\mathrm{AP}) \times \mathcal{P}(-)$ (where AP is a set of atomic propositions), $\Omega = 2$, $\Gamma = \{\bot, \top, \wedge, \vee\}$ with the usual interpretations and $\Lambda = \mathrm{AP} \cup \{\Box, \Diamond\}$ with*

$$\llbracket\, p \in \mathrm{AP} \,\rrbracket_X : \begin{cases} 1 & \to & 2^{\mathcal{P}(\mathrm{AP}) \times \mathcal{P}(X)} \\ * & \mapsto & \{(U, Q) \mid p \in U\} \end{cases}$$

$$\llbracket\, \Box \,\rrbracket_X : \begin{cases} 2^X & \to & 2^{\mathcal{P}(\mathrm{AP}) \times \mathcal{P}(X)} \\ P & \mapsto & \{(U, Q) \mid Q \subseteq P\} \end{cases}$$

$$\llbracket\, \Diamond \,\rrbracket_X : \begin{cases} 2^X & \to & 2^{\mathcal{P}(\mathrm{AP}) \times \mathcal{P}(X)} \\ P & \mapsto & \{(U, Q) \mid P \cap Q \neq \varnothing\} \end{cases}$$

This yields the standard modal logic.

Similarly, take $F = \mathcal{P}(\mathrm{AP}) \times \mathcal{D}_{\leq 1}(-)$, $\Omega = [0,1]$ (with $x \sqsubseteq y \iff x \geq y$), $\Gamma = \{\bot, \top, \min, \max\}$ with the usual interpretations and $\Lambda = \mathrm{AP} \cup \{\mathbb{E}\}$ with

$$\llbracket\, p \in \mathrm{AP} \,\rrbracket_X : \begin{cases} 1 & \to & [0,1]^{\mathcal{P}(\mathrm{AP}) \times \mathcal{D}_{\leq 1} X} \\ * & \mapsto & ((U, \zeta) \mapsto \mathbb{1}_U(p)) \end{cases}$$

$$\llbracket\, \mathbb{E} \,\rrbracket_X : \begin{cases} [0,1]^X & \to & [0,1]^{\mathcal{P}(\mathrm{AP}) \times \mathcal{D}_{\leq 1} X} \\ f & \mapsto & \left((U, \zeta) \mapsto \int f \mathrm{d}\zeta\right) \end{cases}$$

This yields a many-valued logic used to compute probabilities of reaching certain sets of states.

Definition 1.1.6 (semantics of formulæ)**.** *Let Γ and Λ be propositional and modal signatures and let $c : X \to FX$ be an F-coalgebra. A formula φ — with free variables $u_1, \ldots, u_m$ — is assigned its* denotation *over c; it is given by a function*

$$\llbracket\, \varphi \,\rrbracket_c : \left(\Omega^X\right)^m \longrightarrow \Omega^X$$

that is defined inductively in the following way. Here f is a function from X to Ω^m and we write μ and ν for the least and greatest fixed points of monotone functions (whose existence are guaranteed by the Knaster-Tarski theorem).

$$\llbracket\, u_i \,\rrbracket_c (f) = f_i$$

$$\llbracket\, \boxdot_\gamma (\varphi_1, \ldots, \varphi_n) \,\rrbracket_c (f) = \llbracket\, \gamma \,\rrbracket \circ \langle \llbracket\, \varphi_1 \,\rrbracket_c, \ldots, \llbracket\, \varphi_n \,\rrbracket_c \rangle$$

$$\llbracket\, \heartsuit_\lambda (\varphi_1, \ldots, \varphi_n) \,\rrbracket_c (f) = \llbracket\, \lambda \,\rrbracket_X \left(\langle \llbracket\, \varphi_1 \,\rrbracket_c, \ldots, \llbracket\, \varphi_n \,\rrbracket_c \rangle (f) \right) \circ c$$

$$\llbracket\, \mu u. \varphi \,\rrbracket_c (f) = \mu. \left[\llbracket\, \varphi \,\rrbracket_c \left(f_1, \ldots, f_{m-1}, -\right) \right]$$

$$\llbracket\, \nu u. \varphi \,\rrbracket_c (f) = \nu. \left[\llbracket\, \varphi \,\rrbracket_c \left(f_1, \ldots, f_{m-1}, -\right) \right]$$

In the last two clauses it is assumed, by suitably rearranging variables, that the bound fixed point variable u is u_m, the last one among the free variables u_1, $\ldots, u_m$ of φ.

Lemma 1.1.7 (Yoneda for modalities)**.** *There is a bijection $\lambda \mapsto \tau_\lambda$ between n-ary modalities and arrows $\tau : F(\Omega^n) \to \Omega$ such that the mappings $f \mapsto \tau \circ Ff$ are monotone. It is given by*

$$\forall f \in \mathrm{Hom}(X, \Omega)^n, \llbracket\, \lambda \,\rrbracket_X (f) = \tau_\lambda \circ Ff$$

In the following we will treat a modality λ and the arrow τ_λ as the same object.

Example 1.1.8. *The modalities given in Example 1.1.5 can be factored through the following arrows :*

$$\tau_{p \in \mathrm{AP}} : \begin{cases} \mathcal{P}(\mathrm{AP}) \times \mathcal{P}(2) & \to & 2 \\ (U, Q) & \mapsto & \mathbb{1}_U(p) \end{cases}$$

$$\tau_\Box = \forall \circ \pi' : \begin{cases} \mathcal{P}(\mathrm{AP}) \times \mathcal{P}(2) & \to & 2 \\ (U, Q) & \mapsto & \forall x \in Q, x = 1 \end{cases}$$

$$\tau_\Diamond = \exists \circ \pi' : \begin{cases} \mathcal{P}(\mathrm{AP}) \times \mathcal{P}(2) & \to & 2 \\ (U, Q) & \mapsto & \exists x \in Q, x = 1 \end{cases}$$

$$\tau_\mathbb{E} = e \circ \pi' : \begin{cases} \mathcal{P}(\mathrm{AP}) \times \mathcal{D}_{\leq 1}([0,1]) & \to & [0,1] \\ (U, \zeta) & \mapsto & \int_{[0,1]} x \mathrm{d}\zeta(x) \end{cases}$$

1.2 Model Checking Games

Classic modal logic is linked to game semantics via model checking games : computing the semantics of a modal formula amounts to computing the winning positions in well-crafted parity games. When the only fixed point operators involved in the modal formula are greatest fixed points, these parity games can be reduced to safety games. During this internship, these characterization results were extended to coalgebraic modal logic (see Example 2.3.11).

Definition 1.2.1 (parity game). *A* parity game *is played between two players,* Spoiler *(S) and* Duplicator *(D) on a (possibly infinite) graph* $G = (Q = Q_S + Q_D, E)$*, equipped with a priority function* $\mathrm{pr} : Q \to [0, d]$ *for some* $d \in \mathbb{N}$*. Spoiler and Duplicator play according to Table 1.*

A play *is a finite or infinite sequence of positions* $(q_n)_{n\in\mathbb{N}}$ *such that two consecutive positions are always linked :* $(q_n, q_{n+1}) \in E$*. If the play is finite, it is won by Spoiler if the last position is in* Q_D *and by Duplicator if the last position is in* Q_S*. If the play is infinite, take* $p = \limsup_{n\in\mathbb{N}} \mathrm{pr}(q_n)$ *the highest priority that is infinitely visited during the play. If* p *is even, the play is won by Duplicator, and if* p *is odd, the play is won by Spoiler.*

A strategy *for Duplicator is a function* $f : Q_D \to Q$ *such that for all* $q \in Q_D$*,* $(q, f(q)) \in E$*. It is a* winning strategy *for Duplicator from a position* q *if all plays starting from* q *(*$q_0 = q$*) and following this strategy (i.e. such that if* $q_n \in Q_D$*,* q_{n+1} *is defined and equal to* $f(q_n)$*) are won by Duplicator. A position* q *is* winning *for Duplicator if there is a winning strategy from* q *for Duplicator. Strategies for Spoiler and winning conditions are defined similarly.*

Here our definition of a strategy is history-independant. It should not be so, but, since for parity games it can be shown that each winning position is witnessed by a history-independant winning strategy, we directly present them as such. It can also be shown that all positions are either winning for Duplicator or winning for Spoiler.

Safety games are an instance of parity games : they also require for the underlying graph G to be bipartite (Duplicator and Spoiler play alternatively) and for all the positions to have the same priority (a play is won by Duplicator if it is infinite or if Spoiler gets stuck).

Finally, invariants are used to compute the winning positions of safety games : the greatest invariant is the set of winning positions for Duplicator.

Definition 1.2.2 (invariant). *In a safety game an invariant for Duplicator (or trap set for Spoiler) is a set* $I \subseteq Q_S$ *of positions* q *for Spoiler such that for each valid move* $(q, q') \in E$ *for Spoiler, Duplicator can answer with a valid move* $(q', q'') \in E$ *such that* q'' *is in the invariant* I*.*

Although small progress measures allow for the computation of the sets of winning positions of finite parity games [15], in Section 2 we will also be interested in infinite parity games, and we will thus reduce them to well-crafted safety games instead to compute their sets of winning positions.

1.3 Equational Systems and Progress Measures

Small progress measures can be extended to under-approximate nested alternating fixed points, written here as equational systems [13].

1.3.1 Equational Systems

Definition 1.3.1 (equational system). *Let* L_1*, ...,* L_m *be complete lattices. An* equational system *over* $L_1, \dots, L_m$ *is an expression of the form*

$$\begin{aligned} u_1 &=_{\eta_1} f_1(u_1, \dots, u_m) \\ &\vdots \\ u_m &=_{\eta_m} f_m(u_1, \dots, u_m) \end{aligned} \tag{1}$$

where $u_1, \dots, u_m$ *are* variables*,* $\eta_1, \dots, \eta_m \in \{\mu, \nu\}$ *are fixed point operators and* $f_1 : L_1 \times \dots \times L_m \to L_1, \dots, f_m : L_1 \times \dots \times L_m \to L_m$ *are monotone functions.*

When $\eta_i = \mu$*,* u_i *is said to be a* μ-variable*, and when* $\eta_i = \nu$*,* u_i *is said to be a* ν*-variables.*

We say that u_i *has bigger priority than* u_j *if* $j < i$*.*

An equational system is nothing else than a formalism for writing formulæ involving nested alternating

Position	Player	Possible moves	Priority
$q \in Q_S$	S	$q' \in Q$ such that $(q, q') \in E$	$\mathrm{pr}(q)$
$q \in Q_D$	D	$q' \in Q$ such that $(q, q') \in E$	$\mathrm{pr}(q)$

Table 1: Priorities and possible moves in a parity game

fixed points : the solution of an equational system is therefore defined as the semantics of the corresponding nested alternating fixed point.

Example 1.3.2. *Consider the following equational system :*

$$u_1 =_{\eta_1} f_1(u_1, u_2)$$
$$u_2 =_{\eta_2} f_2(u_1, u_2)$$

Its solution is the semantics of the following formula (and of its subformulæ) :

$$\eta_2 u_2.f_2(\eta_1 u_1.f_1(u_1, u_2), u_2)$$

Equational systems share a strong link with parity games, as originally shown in Appendix A of [13].

Proposition 1.3.3 (equational systems for finite parity games)**.** *Consider a parity game on a finite graph $G = (Q, E)$ with priority function $\mathrm{pr} : Q \to [1, d]$. For each priority $i \in [1, d]$, write n_i for the number of positions in Q with priority i, and enumerate Q as $q_{1,1}, \dots, q_{1,n_1}, \dots, q_{d,1}, \dots, q_{d,n_d}$ so that $\mathrm{pr}(q_{i,j}) = i$. The equational system*

$$u_1 =_{\eta_1} f_1(u_1, \dots, u_d)$$
$$\dots$$
$$u_d =_{\eta_d} f_d(u_1, \dots, u_d)$$

given by

- $u_i \in 2^{n_i}$,
- *η_i is ν if i is even, μ if it is odd,*
- *$f_i : 2^{n_1} \times \dots \times 2^{n_d} \to 2^{n_i}$ is defined, for each $j \in [1, n_i]$, by*

$$j \in f_i(u_1, \dots, u_d) \iff \begin{cases} \exists (q_{i,j}, q_{i',j'}) \in E \mid j' \in u_{i'} & \text{if } q_{i,j} \in Q_D \\ \forall (q_{i,j}, q_{i',j'}) \in E \mid j' \in u_{i'} & \text{if } q_{i,j} \in Q_S \end{cases}$$

where $\pi_{j'} : 2^{n_{i'}} \to 2$ is the projection on the j'-th coordinate and $\sqcup$ and $\sqcap$ are respectively the supremum and the infimum in the complete lattice 2,

characterizes the winning positions of the parity game. Indeed, $q_{i,j}$ is winning for D if and only if $\pi_j(l_i^{\mathrm{sol}}) = 1$.

1.3.2 Progress Measures

Recall that small progress measures characterize the winning positions of finite parity games. Likewise, (lattice-theoretic) progress measures characterize the under-approximants of the solutions of equational systems.

These progress measures merge the notions of post-fixed points, which under-approximate greatest fixed points by the Knaster-Tarski theorem, and of iterates of $\perp$, which under-approximate least fixed points by the Cousot-Cousot theorem. To keep track of multiple of these iterates for multiple least fixed points, progress measures use the notion of prioritized ordinals.

Definition 1.3.4 (prioritized ordinal)**.** *Let E be Equational System* (1)*. Write $1 \leq i_1 < \dots < i_k \leq n$ for those indices i such that $\eta_i = \mu$. A* prioritized ordinal *for E is then a k-tuple $(\alpha_1, \dots, \alpha_k)$ of ordinals.*

For $i \in [1, m]$, the preorder $\preceq_i$ between prioritized ordinals is defined as follows. Find $a \in [1, k]$ such that $i_{a-1} < i \leq i_a$. Then $(\alpha_1, \dots, \alpha_k) \preceq_i (\alpha'_1, \dots, \alpha'_k)$ if and only if $(\alpha_a, \dots, \alpha_k) \preceq (\alpha'_a, \dots, \alpha'_k)$ with $\preceq$ the lexicographic order for which the last element of a tuple is the most important.

Similarly, $=_i$ holds when $\preceq_i$ holds both ways, and $\prec_i$ holds when $\preceq_i$ holds but not $=_i$.

As the original definition of progress measures did not fit exactly our needs for Sections 2 and 3, a generalized version was designed to do so. It is this new version that is directly presented here.

Definition 1.3.5 (progress measure)**.** *Let E being Equational System* (1)*. A* progress measure *for E is a tuple*

$$p = ((\overline{\alpha}_1, \ldots, \overline{\alpha}_k), (p_i(\alpha_1, \ldots, \alpha_k))_{i,\alpha_1,\ldots,\alpha_k})$$

with $(\overline{\alpha}_1, \ldots, \overline{\alpha}_k)$ the maximum prioritized ordinal *and $p_i(\alpha_1, \ldots, \alpha_k)$ the* approximants *for each $i \in [1, m]$ and $\alpha_1 \leq \overline{\alpha}_1, \ldots, \alpha_k \leq \overline{\alpha}_k$.*

The approximants are subject to the following conditions, where all the ordinals considered are bounded by one of $\overline{\alpha}_1, \ldots, \overline{\alpha}_k$, according to their indices in the corresponding prioritized ordinal.

1. ***(monotonicity)*** *Given $i \in [1, m]$, if $(\alpha_1, \ldots, \alpha_k) \preceq_i (\alpha'_1, \ldots, \alpha'_k)$, then (in L_i)*

$$p_i(\alpha_1, \ldots, \alpha_k) \sqsubseteq p_i(\alpha'_1, \ldots, \alpha'_k)$$

2. ***(μ-variables)*** *If $i \in [1, m]$ is such that $\eta_i = \mu$, then for all prioritized ordinals $(\alpha_1, \ldots, \alpha_k)$,*

$$p_i(\alpha_1, \ldots, \alpha_k) \sqsubseteq f_i \begin{pmatrix} \bigsqcup_{(\delta_1,\ldots,\delta_k) \prec_i (\alpha_1,\ldots,\alpha_k)} p_1(\delta_1, \ldots, \delta_k) \\ \cdots \\ \bigsqcup_{(\delta_1,\ldots,\delta_k) \prec_i (\alpha_1,\ldots,\alpha_k)} p_m(\delta_1, \ldots, \delta_k) \end{pmatrix}$$

3. ***(ν-variables)*** *If $i \in [1, m]$ is such that $\eta_i = \nu$, then for all prioritized ordinals $(\alpha_1, \ldots, \alpha_k)$,*

$$p_i(\alpha_1, \ldots, \alpha_k) \sqsubseteq f_i \begin{pmatrix} \bigsqcup_{(\delta_1,\ldots,\delta_k) \preceq_i (\alpha_1,\ldots,\alpha_k)} p_1(\delta_1, \ldots, \delta_k) \\ \cdots \\ \bigsqcup_{(\delta_1,\ldots,\delta_k) \preceq_i (\alpha_1,\ldots,\alpha_k)} p_m(\delta_1, \ldots, \delta_k) \end{pmatrix}$$

Progress measures characterize the solutions of equational systems by under-approximating them.

Theorem 1.3.6 ([13])**.** *Let E be Equational System* (1) *and let $\left(l_1^{\mathrm{sol}}, \ldots, l_m^{\mathrm{sol}}\right)$ be its solution. Then*

- ***(soundness)*** *a progress measure $((\overline{\alpha}_1, \ldots, \overline{\alpha}_k), (p_i(\alpha_1, \ldots, \alpha_k)_{i,\alpha_1,\ldots,\alpha_k}))$ gives a lower bound of the solution, that is*

$$p_i(\overline{\alpha}_1, \ldots, \overline{\alpha}_k) \sqsubseteq l_i^{\mathrm{sol}}$$

  for all $i \in [1, m]$;

- ***(completeness)*** *there are ordinals $\overline{\alpha}_1, \ldots, \overline{\alpha}_k$, bounded by the supremum of the lenghts of strictly ascending chains in $L_1, \ldots, L_m$, for which there is a progress measure $((\overline{\alpha}_1, \ldots, \overline{\alpha}_k), (p_i(\alpha_1, \ldots, \alpha_k)_{i,\alpha_1,\ldots,\alpha_k}))$ that achieves the solution, that is*

$$p_i(\overline{\alpha}_1, \ldots, \overline{\alpha}_k) = l_i^{\mathrm{sol}}$$

  for all $i \in [1, m]$.

1.4 $\mathbf{CLat}_\sqcap$-fibrations and Codensity Liftings

The previous concepts will be used in the setting of $\mathbf{CLat}_\sqcap$-fibrations and with monotone functions built from codensity liftings [11].

Definition 1.4.1 ($\mathbf{CLat}_\sqcap$-fibration)**.** *Given two categories $\mathbb{C}$ (the* total category*) and $\mathbb{B}$ (the* base category*), a* $\mathbf{CLat}_\sqcap$-fibration *is a functor $p : \mathbb{C} \to \mathbb{B}$ such that*

- *the* fiber *$\mathbb{C}_X$ (the subcategory of $\mathbb{C}$ with objects those E such that $pE = X$ and arrows those f such that $pf = \mathrm{id}_X$) is a complete lattice ;*

- *for any $f : X \to Y$ in $\mathbb{B}$ and $Q \in \mathbb{C}_Y$, there is an object $f^*Q \in \mathbb{C}_X$ and a $\mathbb{C}$-arrow $\overline{f}Q : f^*Q \to Q$ (the* cartesian lifting *of f and Q) such that for any $P \in \mathbb{C}_X$ and $g : P \to Q$ in E, if $pg = f$ then there is a unique $h : P \to f^*Q$ such that $g = \overline{f}(Q) \circ h$ and $ph = \mathrm{id}_X$, as depicted in Diagram* (2)*.*

$$\begin{tikzcd}
\mathbb{C} \arrow[dd, "p"] & f^*Q \arrow[r, "\overline{f}(Q)"] & Q \\
& P \arrow[u, dashed, "h"] \arrow[ur, "g"] & \\
\mathbb{B} & X \arrow[r, "f=pg"] \arrow[loop below, "\mathrm{id}=ph"] & Y
\end{tikzcd} \qquad (2)$$

- *the correspondances $-^*$ and $\overline{-}$ are functorial :*

$$\mathrm{id}_Y^* Q = Q \qquad (g \circ f)^* Q = f^*(g^* Q)$$

$$\overline{\mathrm{id}_Y}(Q) = \mathrm{id}_Q \quad \overline{g \circ f}(Q) = \overline{g}(Q) \circ \overline{f}(g^* Q)$$

In particular, for $f : X \to Y$ in $\mathbb{B}$, the pullback functor *$f^* : \mathbb{C}_Y \to \mathbb{C}_X$ preserves all meets $\bigsqcap$.*

Definition 1.4.2 (decent arrow)**.** *Given two objects P and Q in the total category $\mathbb{C}$ of a fibration $p : \mathbb{C} \to \mathbb{B}$, an arrow $f : pP \to pQ$ is* decent *from P to Q if and only if there is an arrow $\widetilde{f} : P \to Q$ in $\mathbb{C}$ such that $p\widetilde{f} = f$. Since p is a $\mathbf{CLat}_\sqcap$-fibration, this amounts to say that $P \sqsubseteq f^*Q$ in $\mathbb{C}_X$. We write $f : P \dot{\to} Q$.*

For $f : X \to Y$ in $\mathbb{C}$ and $Q \in \mathbb{C}_Y$, f^*Q is the greatest element of $\mathbb{C}_X$ from which f is decent to Q.

Example 1.4.3. *In this work we will focus on two specific families of $\mathbf{CLat}_\sqcap$-fibrations, both parameterized by a complete lattice Ω.*

- *Consider the slice category $(\mathbf{Set} \downarrow \Omega)$; it is a fibration and its fibers $\mathrm{Hom}(X, \Omega)$ can be ordered by extending the order on Ω. The category $\mathbf{Pred}_\Omega$ of Ω-predicates is the subcategory of $(\mathbf{Set} \downarrow \Omega)$ with the same objects and with arrows between two objects $f : X \to \Omega$ and $g : Y \to \Omega$ those functions $k : X \to Y$ such that $f \sqsubseteq g \circ k$.*

  The composite functor $p : \mathbf{Pred}_\Omega \to (\mathbf{Set} \downarrow \Omega) \to \mathbf{Set}$ sends an object $f : X \to \Omega$ to its codomain X and an arrow $k : X \to Y$ to the function k. Since Ω is a complete lattice, $\mathrm{Hom}(X, \Omega)$ is one too and thus p is a $\mathbf{CLat}_\sqcap$-fibration.

  The cartesian lifting of $k : X \to Y$ and $f \in \mathrm{Hom}(Y, \Omega)$ is simply $k : f \circ k \to f$.

- *Pulling back $\mathbf{Pred}_\Omega$ along the diagonal functor $\Delta : \mathbf{Set} \to \mathbf{Set}$ (which sends a set X to $X \times X$) as in Diagram (3) yields a change-of-base $\mathbf{CLat}_\sqcap$-fibration $\mathbf{ERel}_\Omega \to \mathbf{Set}$.*

$$\begin{tikzcd}
\mathbf{ERel}_\Omega \arrow[r, dashed] \arrow[d, dashed] & \mathbf{Pred}_\Omega \arrow[d] \\
\mathbf{Set} \arrow[r, "\Delta"] & \mathbf{Set}
\end{tikzcd} \tag{3}$$

  The category $\mathbf{ERel}_\Omega$ of Ω-(endo)relations has objects functions $r : X^2 \to \Omega$ and arrows between two objects $r : X^2 \to \Omega$ and $s : Y^2 \to \Omega$ those functions $k : X \to Y$ such that $r \sqsubseteq s \circ \Delta k$.

  The cartesian lifting of an arrow $k : X \to Y$ and an object $r : Y^2 \to \Omega$ is simply $k : r \circ \Delta k \to r$.

When $\Omega = 2$, these two fibrations are the classic $\mathbf{Pred} \to \mathbf{Set}$ and $\mathbf{ERel} \to \mathbf{Set}$ fibrations of predicates and endorelations over X.

Definition 1.4.4 (codensity lifting $F^{\mathbf{\Omega},\tau}$)**.** *Let $F : \mathbb{B} \to \mathbb{B}$ be an endofunctor on the base category of a $\mathbf{CLat}_\sqcap$-fibration $p : \mathbb{C} \to \mathbb{B}$. The choice of a family of arrows $\tau = (\tau_\sigma : F\Omega_\sigma \to \Omega_\sigma)_{\sigma \in \Sigma}$ in $\mathbb{B}$ and of a family of objects $\mathbf{\Omega} = (\mathbf{\Omega}_\sigma \in \mathbb{C}_{\Omega_\sigma})_{\sigma\in\Sigma}$ yields the* codensity lifting *$F^{\mathbf{\Omega},\tau} : \mathbb{C} \to \mathbb{C}$ of F with parameters $(\mathbf{\Omega}, \tau)$, defined as follows. For $P \in \mathbb{C}_X$,*

$$F^{\mathbf{\Omega},\tau} P = \bigsqcap_{\sigma\in\Sigma} \bigsqcap_{k : P \dot{\to} \mathbf{\Omega}_\sigma} (\tau_\sigma \circ Fk)^* \mathbf{\Omega}_\sigma$$

and for $f : P \to Q$, $F^{\mathbf{\Omega},\tau} f$ is the composite

$$F^{\mathbf{\Omega},\tau} P \sqsubseteq (F(pf))^* \left(F^{\mathbf{\Omega},\tau} Q\right) \xrightarrow{\overline{F(pf)}\left(F^{\mathbf{\Omega},\tau}Q\right)} F^{\mathbf{\Omega},\tau} Q$$

Example 1.4.5. *In this work we will be focusing on the following codensity liftings*

- *In $\mathbf{Pred}_\Omega$, take $\tau : F\Omega \to \Omega$ a unary modality over F. Then for $f : X \to \Omega$,*

$$\begin{aligned} F^{\mathrm{id}_\Omega,\tau} f &= \bigsqcap_{\substack{k : X \to \Omega \\ f \sqsubseteq \mathrm{id}_\Omega \circ k}} \mathrm{id}_\Omega \circ \tau \circ Fk \\ &= \tau \circ Ff \\ &= [\![\ \tau\ ]\!]_X\,(f) \end{aligned}$$

  since $k \mapsto \tau \circ Fk$ is monotone.

- *In $\mathbf{ERel}_\Omega$, take $\mathbf{\Omega} : \Omega^2 \to \Omega$ and $\tau : F\Omega \to \Omega$ a unary modality over F. Then for $r : X^2 \to \Omega$,*

$$\begin{aligned} F^{\mathbf{\Omega},\tau} r &= \bigsqcap_{\substack{k : X \to \Omega \\ r \sqsubseteq \mathbf{\Omega} \circ \Delta k}} \mathbf{\Omega} \circ \Delta(\tau \circ Fk) \\ &= \bigsqcap_{\substack{k : X \to \Omega \\ r \sqsubseteq \mathbf{\Omega} \circ \Delta k}} \mathbf{\Omega} \circ \Delta\big([\![\ \tau\ ]\!]_X\,(k)\big) \end{aligned}$$

  For example when $\Omega = 2$ is the boolean lattice, $F = \mathcal{P}$, $\mathbf{\Omega} = \Leftrightarrow$ is the equality relation and $\tau = \exists$ is the may-modality, two subsets P and Q

of X are related by $F^{\Leftrightarrow,\exists}r$ if and only if for any function $k : X \to 2$ that preserves the equivalence classes of r^{eq} (the equivalence closure of r),

$$\exists x \in P, k(x) = 1 \iff \exists y \in Q, k(y) = 1$$

2 Codensity Parity Games

Once these preliminaries were understood, it was then possible to go through the article that introduced the categorical link between bisimulation notions and games [12] and generalize each step of its development to nested alternating fixed points.

2.1 Codensity Safety Games

The following is therefore a brief account of the content of this article, which will be mimicked and extended in Sections 2.2 and 2.3.

Assume the setting of Definition 1.4.4 by taking $F : \mathbb{B} \to \mathbb{B}$ an endofunctor on the base category of a $\mathbf{CLat}_\sqcap$-fibration $p : \mathbb{C} \to \mathbb{B}$ and $\tau_\sigma : F\Omega_\sigma \to \Omega_\sigma$, $\mathbf{\Omega}_\sigma \in \mathbb{C}_{\Omega_\sigma}$ for $\sigma \in \Sigma$. Consider an F-coalgebra $c : X \to FX$ as well.

Definition 2.1.1 (single-variable predicate transformer $\Phi_c^{\mathbf{\Omega},\tau}$)**.** *The* predicate transformer $\Phi_c^{\mathbf{\Omega},\tau} : \mathbb{C}_X \to \mathbb{C}_X$ *is the monotone function defined by*

$$\begin{aligned}\Phi_c^{\mathbf{\Omega},\tau} P &= c^* \left(F^{\mathbf{\Omega},\tau} P\right) \\ &= \prod_{\sigma \in \Sigma} \prod_{k : P \dot\to \mathbf{\Omega}_\sigma} (\tau_\sigma \circ Fk \circ c)^* \mathbf{\Omega}_\sigma\end{aligned}$$

Definition 2.1.2 (codensity bisimulations and bisimilarities)**.** *The post-fixed points of $\Phi_c^{\mathbf{\Omega},\tau}$ are called* codensity bisimulations *over c. The greatest codensity bisimulation is called the* codensity bisimilarity *over c and is written $\nu\Phi_c^{\mathbf{\Omega},\tau}$ (it is the greatest fixed point of $\Phi_c^{\mathbf{\Omega},\tau}$ by the Knaster-Tarski theorem).*

The concept of codensity bisimilarity is expressive enough to instantiate the classic bisimulation relation [16] as well as the more recent bisimulation distance [17].

Example 2.1.3. *The codensity liftings of Example 1.4.5 yield codensity bisimulations and codensity bisimilarities.*

- *In $\mathbf{Pred}_\Omega$, take $\tau : F\Omega \to \Omega$ a unary modality over F. Then, for $f : X \to \Omega$,*

$$\Phi_c^{\mathrm{id}_\Omega,\tau} f = [\![\ \tau\ ]\!]\,(f) \circ c = [\![\ \heartsuit_\tau(u)\ ]\!]_c\,(f) \quad (4)$$

  The codensity bisimilarity is thus

$$\nu\Phi_c^{\mathrm{id}_\Omega,\tau} = [\![\ \nu u.\heartsuit_\tau(u)\ ]\!]_c$$

- *In $\mathbf{ERel}_\Omega$, take $\mathbf{\Omega} : \Omega^2 \to \Omega$ and $\tau : F\Omega \to \Omega$ a unary modality over F. Then for $r : X^2 \to \Omega$,*

$$\begin{aligned}\Phi_c^{\mathbf{\Omega},\tau} r &= \prod_{\substack{k : X \to \Omega \\ r \sqsubseteq \mathbf{\Omega} \circ \Delta k}} \mathbf{\Omega} \circ \Delta\big([\![\ \tau\ ]\!]\,(k) \circ c\big) \\ &= \prod_{\substack{k : X \to \Omega \\ r \sqsubseteq \mathbf{\Omega} \circ \Delta k}} \mathbf{\Omega} \circ \Delta\big([\![\ \heartsuit_\tau\ ]\!]_c\,(k)\big)\end{aligned}$$

  For example when $\Omega = 2$ is the boolean lattice, $F = \mathcal{P}$ is the powerset functor, $\mathbf{\Omega} = \Leftrightarrow$ is the equality relation and $\tau = \exists$ is the may-modality, two states x and y in X are related by $\Phi_c^{\Leftrightarrow,\exists} r$ if and only if for any predicate $k : X \to 2$ that is constant on the equivalence classes of r^{eq}, $[\![\ \Diamond\ ]\!]_c\,(k)$ (or $\Diamond k$ with the notation of classic modal logic) is true on x if and only if it is true on y :

$$\exists x' \in c(x), k(x') = 1 \iff \exists y' \in c(y), k(y') = 1$$

  It is easy to see that x and y are related by $\Phi_c^{\Leftrightarrow,\exists} r$ if and only if each element of $c(x)$ (respectively $c(y)$) is related by r^{eq} to an element of $c(y)$ (respectively $c(x)$). One will thus recognize that codensity bisimulations are the classic bisimulations over the Kripke frame given by c and that, similarly, the codensity bisimilarity is the bisimilarity relation over this Kripke frame [16].

  Similarly, taking $\Omega = ([0,1], \geq)$, $F = \mathcal{D}_{\leq 1}$ the subdistribution functor, $\mathbf{\Omega} = d_{[0,1]}$ the euclidean distance on $[0,1]$ and $\tau = e$ as introduced in Example 1.1.8, the codensity bisimilarity is the usual bisimulation distance for the transition function $c : X \to \mathcal{D}_{\leq 1} X$ of a Markov chain [17] : for $d : X^2 \to [0,1]$,

$$\Phi_c^{d_{[0,1]},e}(d)(x,y) = \sup_{\substack{f : X \to [0,1] \\ ||f||_d \leq 1}} \left|\mathbb{E}_{c(x)}[f] - \mathbb{E}_{c(y)}[f]\right|$$

- *For an example with non-trivial Σ, take $F = 2 \times -^{\Sigma}$ as for a deterministic automaton, $\tau_\sigma(b, f) = f(\sigma)$ for $\sigma \in \Sigma$ and $\tau_\epsilon(b, f) = b$ with $\mathbf{\Omega}_\sigma = \mathbf{\Omega}_\epsilon$ the equality relation on $\Omega = 2$. The resulting codensity bisimilarity relates states for which the automaton recognizes the same language when they are the initial states.*

Codensity bisimilarities can be characterized by the winning positions of the following game. In particular, since codensity bisimilarities are greatest fixed points, this game is a safety game.

Definition 2.1.4 (untrimmed codensity safety game)**.** *The* untrimmed codensity safety game *(originally the untrimmed codensity game for bisimilarity) is the safety game shown in Table 2.*

The proof of this result makes use of joint codensity bisimulations.

Definition 2.1.5 (joint codensity bisimulation)**.** *A subset $\mathcal{V} \subseteq \mathbb{C}_X$ is a* joint codensity bisimulation *over c if $\bigsqcup_{P \in \mathcal{V}} P$ is a codensity bisimulation over c. The* joint codensity bisimilarity *is the downset $\downarrow \left(\nu\Phi_c^{\mathbf{\Omega},\tau}\right) = \left\{P \in \mathbb{C}_X \mid P \sqsubseteq \nu\Phi_c^{\mathbf{\Omega},\tau}\right\}$.*

Theorem 2.1.6. *The invariants of the untrimmed codensity safety game are exactly the joint codensity bisimulations. As such, the set of winning positions of this game, as its greatest invariant, is exactly the joint codensity bisimilarity $\downarrow \left(\nu\Phi_c^{\mathbf{\Omega},\tau}\right)$.*

Although the previous theorem gives the expected categorical link between bisimulation notions and game theory, the resulting game lacks practicality as its set of positions can be very large. Fortunately, it is possible to trim it by using generating sets.

Definition 2.1.7 (generating set)**.** *A* generating set *of the fiber $\mathbb{C}_X$ is a set $\mathcal{G}$ such that for all $P \in \mathbb{C}_X$, $P = \bigsqcup\{Q \in \mathcal{G} \mid Q \sqsubseteq P\}$.*

Example 2.1.8. *In this work we will be using the two following families of generating sets both parameterized by a complete lattice Ω, generated by elements $\omega \in \Omega$, and a set X.*

- *The complete lattice of Ω-predicates over X is generated by the set of those functions*

$$\delta_x^\omega : \begin{cases} X & \to & \Omega \\ x' & \mapsto & \begin{cases} \omega & \text{if } x = x' \\ \bot & \text{otherwise} \end{cases} \end{cases}$$

  for $x \in X$. When $\Omega = 2$, this set is simply the set of singletons in $\mathcal{P}(X)$.

- *The complete lattice of Ω-relations over X is generated by the set of those functions*

$$\delta_{x,y}^\omega : \begin{cases} X^2 & \to & \Omega \\ x', y' & \mapsto & \begin{cases} \omega & \text{if } x = x' \wedge y = y' \\ \bot & \text{otherwise} \end{cases} \end{cases}$$

  When $\Omega = 2$, this is simply the set of singletons in $\mathcal{P}\left(X^2\right)$.

We then get a smaller game, still characterizing codensity bisimilarities, that is reminescent of the classic bisimulation game [7] and instantiates a new game characterizing the bisimulation distance.

Definition 2.1.9 (trimmed codensity safety game)**.** *Let $\mathcal{G}$ be a generating set of the fiber $\mathbb{C}_X$. The* trimmed codensity safety game *is the same game as the (untrimmed) codensity safety game (Definition 2.1.4), except D can only play moves in $\mathcal{G}$ (instead of $\mathbb{C}_X$).*

Theorem 2.1.10. *The invariants of the trimmed codensity safety game are exactly the joint codensity bisimulations included in $\mathcal{G}$. As such, the set of winning positions of this game, as its greatest invariant, is exactly $\downarrow \left(\nu\Phi_c^{\mathbf{\Omega},\tau}\right) \cap \mathcal{G}$.*

Finally, codensity bisimilarity can also be transfered between different fibrations linked by fibered functors verifying specific conditions.

Theorem 2.1.11 (Transfer of codensity bisimilarity)**.** *Let $q : \mathbb{D} \to \mathbb{B}$ be another $\mathbf{CLat}_\sqcap$-fibration and let $T : \mathbb{C} \to \mathbb{D}$ be a full fibered functor from p to q preserving fibered meets. Then, $(T\mathbf{\Omega} = (T\mathbf{\Omega}_\sigma)_{\sigma \in \Sigma}, \tau)$ parameterizes a codensity lifting of F along q, and $\nu\Phi_c^{T\mathbf{\Omega},\tau} = T(\nu\Phi_c^{\mathbf{\Omega},\tau})$.*

Position	Player	Possible moves
$P \in \mathbb{C}_X$	S	$\sigma \in \Sigma$, $k : X \to \Omega_\sigma$ such that $P \not\sqsubseteq (\tau_\sigma \circ Fk \circ c)^* \mathbf{\Omega}_\sigma$
$k : X \to \Omega_\sigma$	D	$P \in \mathbb{C}_X$ such that $P \not\sqsubseteq k^* \mathbf{\Omega}_\sigma$

Table 2: Untrimmed codensity safety game

2.2 Multivariate Predicate Transformer

To extend the previous results to nested alternating fixed points, it was first necessary to define a predicate transformer that would take into account multiple variables.

One way to do so is to realize that the original predicate transformer instantiates the semantics of a unary modality (Equation (4)) because when the parameter τ and Ω of the codensity lifting are chosen to be a monotone function and a complete lattice, τ ends up being a unary modality (Lemma 1.1.7).

Since the Henessy-Milner theorem gives a deep relationship between bisimulations and modal logic, it is thus sensible to set out to preserve this result when extending predicate transformers to multiple variables. This is the choice that was made with the following definitions.

Definition 2.2.1 (multivariate codensity lifting)**.** *Let $F : \mathbb{B} \to \mathbb{B}$ be an endofunctor on the base category of a fibration $p : \mathbb{C} \to \mathbb{B}$ for which $\mathbb{B}$ has finite products. The choice of a family of arrows $(\tau_\sigma : F(\Omega_\sigma^n) \to \Omega_\sigma)_{\sigma\in\Sigma}$ in $\mathbb{B}$ and of a family of objects $(\mathbf{\Omega}_\sigma \in \mathbb{C}_{\Omega_\sigma})_{\sigma\in\Sigma}$ yields the* (n-variable) codensity lifting *$F^{\mathbf{\Omega},\tau} : \mathbb{C}^n \to \mathbb{C}$ of F with parameter $(\mathbf{\Omega}, \tau)$, defined as follows. For $P = (P_1, \ldots, P_n) \in \mathbb{C}_{X_1} \times \cdots \times \mathbb{C}_{X_n}$,*

$$F^{\mathbf{\Omega},\tau} P = \bigsqcap_{\sigma\in\Sigma} \bigsqcap_{\substack{k_1 : P_1 \dot\to \mathbf{\Omega}_\sigma \\ \cdots \\ k_n : P_n \dot\to \mathbf{\Omega}_\sigma}} (\tau_\sigma \circ F(k_1 \times \cdots \times k_n))^* \mathbf{\Omega}_\sigma$$

(it is an object of $\mathbb{C}_{F(X_1\times\cdots\times X_n)}$) and for $f : P \to Q$ in $\mathbb{C}^n$, $F^{\mathbf{\Omega},\tau} f$ is the composite

$$F^{\mathbf{\Omega},\tau} P \sqsubseteq (F(p^n f))^* \left(F^{\mathbf{\Omega},\tau} Q\right) \xrightarrow{\overline{F(p^n f)}\left(F^{\mathbf{\Omega},\tau} Q\right)} F^{\mathbf{\Omega},\tau} Q$$

Example 2.2.2. *In $\mathbf{Pred}_\Omega$, take $\tau : F(\Omega^n) \to \Omega$ an n-ary modality. Then for $f_1, \ldots, f_n : X \to \Omega$,*

$$\begin{aligned} F^{\mathrm{id}_\Omega,\tau}(f_1, \ldots, f_n) &= \bigsqcap_{\substack{k : X \to \Omega^n \\ f_1 \sqsubseteq \mathrm{id}_\Omega \circ k_1 \\ \cdots \\ f_n \sqsubseteq \mathrm{id}_\Omega \circ k_n}} \mathrm{id}_\Omega \circ \tau \circ Fk \\ &= \tau \circ F(f_1 \times \cdots \times f_n) \end{aligned}$$

since $k \mapsto \tau \circ Fk$ is monotone. Therefore if $\langle \mathrm{id}_X, \ldots, \mathrm{id}_X\rangle : X \to X^n$,

$$F^{\mathrm{id}_\Omega,\tau}(f_1, \ldots, f_n) \circ F\langle \mathrm{id}_X, \ldots, \mathrm{id}_X\rangle = \llbracket\ \tau\ \rrbracket(\langle f_1, \ldots, f_n\rangle)$$

Assume the setting of Definition 2.2.1, and let $c : X \to FX$ be an F-coalgebra.

Definition 2.2.3 (multivariate predicate transformer)**.** *The* (n-variable) predicate transformer *$\Phi_c^{\mathbf{\Omega},\tau} : (\mathbb{C}_X)^n \to \mathbb{C}_X$ is the function defined by*

$$\begin{aligned} &\Phi_c^{\mathbf{\Omega},\tau}(P_1, \ldots, P_n) = \\ &\quad (F\langle \mathrm{id}_X, \ldots, \mathrm{id}_X\rangle \circ c)^* (F^{\mathbf{\Omega},\tau}(P_1, \ldots, P_n)) \\ &\quad = \bigsqcap_{\sigma\in\Sigma} \bigsqcap_{\substack{k : X \to \Omega_\sigma^n \\ k_1 : P_1 \dot\to \mathbf{\Omega}_\sigma \\ \cdots \\ k_n : P_n \dot\to \mathbf{\Omega}_\sigma}} (\tau_\sigma \circ Fk \circ c)^* \mathbf{\Omega}_\sigma \end{aligned}$$

with $\langle \mathrm{id}_X, \ldots, \mathrm{id}_X\rangle : X \to X^n$ and where k_i denotes the i-th projection of $k : X \to \Omega_\sigma^n$.

Proposition 2.2.4 (monotonicity)**.** *$\Phi_c^{\mathbf{\Omega},\tau}$ is monotone.*

Example 2.2.5. *In $\mathbf{Pred}_\Omega$, take $\tau : F(\Omega^n) \to \Omega$ an n-ary modality over F. Then, for $f : X \to \Omega^n$,*

$$\Phi_c^{\mathrm{id}_\Omega,\tau}(f) = \llbracket\ \tau\ \rrbracket(f) \circ c = \llbracket\ \heartsuit_\tau(u_1, \ldots, u_n)\ \rrbracket_c(f)$$

Example 2.2.5 is actually an instance of a more general result : the multivariate predicate transformer accomodates the semantics of any $\mathbf{C}\mu_{\Gamma,\Lambda}$ formula.

Lemma 2.2.6. *Assume F to be a* **Set***-endofunctor, and let Γ and Λ be propositional and modal signatures over F with a complete lattice of truth values Ω. Then for any $\mathbf{C}\mu_{\Gamma,\Lambda}$ formula ϕ with free variables $u_1, \ldots, u_m$ consisting only of propositional connectives and modalities over F, (that is with no fixed point operator) there is a functor $P(F)$ monomial in F, a $P(F)$-coalgebra $d : X \to P(F)X$ and an arrow $\tau : P(F)(\Omega^m) \to \Omega$ such that*

- *the mappings $f \mapsto \tau \circ P(F) f$ are monotone ;*
- *for any $f : X \to \Omega^m$*

$$[\![\phi]\!]_c (f) = \tau \circ P(F) f \circ d$$

In other words, there is a modality τ over $P(F)$ such that

$$[\![\heartsuit_\tau (u_1, \ldots, u_m)]\!]_d = [\![\phi]\!]_c$$

Proposition 2.2.7. *Assume the setting of Lemma 2.2.6. Then in $\mathbf{Pred}_\Omega \to \mathbf{Set}$,*

$$\Phi_d^{\mathrm{id}_\Omega,\tau} = [\![\phi]\!]_c$$

The same result also holds true when replacing the monomial functor $P(F)$ with the cofree comonad of F. As this additionnal result is not of any use in the following, it is not developed here.

2.3 Codensity Equational Systems and their Parity Games

With multivariate predicate transformers it was now possible to design nested alternating fixed points involving codensity liftings. The goal was then to characterize these fixed points with well-crafted games.

Codensity Equational Systems. As explained in Section 1.3.1, we write nested alternating fixed points under the guise of equational systems.

Let thus $m \in \mathbb{N}$ be an integer, $\mathbb{C} \to \mathbb{B}$ a $\mathbf{CLat}_\sqcap$-fibration and X an object of $\mathbb{B}$. For each $i \in [1, m]$, choose :

- an endofunctor $F_i : \mathbb{B} \to \mathbb{B}$;
- an F_i-coalgebra $c_i : X \to F_i X$;
- a family of labels Σ_i ;
- families of arrows

$$\tau_i = \left(\tau_{i,\sigma} : F_i \left((\Omega_{i,\sigma})^m\right) \to \Omega_{i,\sigma}\right)_{\sigma \in \Sigma_i}$$

- families of objects

$$\mathbf{\Omega}_i = \left(\mathbf{\Omega}_{i,\sigma} \in \mathbb{C}_{\Omega_{i,\sigma}}\right)_{\sigma \in \Sigma_i}$$

- a fixpoint operator $\eta_i \in \{\nu, \mu\}$.

Definition 2.3.1 (codensity equational system)**.** *The* codensity equational system *E, computed in $\mathbb{C} \to \mathbb{B}$, is Equational System* (5)*.*

$$\begin{aligned} u_1 &=_{\eta_1} \Phi_{c_1}^{\mathbf{\Omega}_1,\tau_1} (u_1, \ldots, u_m) \\ &\vdots \\ u_m &=_{\eta_m} \Phi_{c_m}^{\mathbf{\Omega}_m,\tau_m} (u_1, \ldots, u_m) \end{aligned} \tag{5}$$

We write $\left(l_1^{\mathrm{sol}}, \ldots, l_m^{\mathrm{sol}}\right)$ for its solution.

Let $i_1 < \cdots < i_k$ be the indices of the μ-variables, and, for $j \in [1, m]$ such that $\eta_j = \mu$, write a_j for the $a \in [1, k]$ such that $i_{a_j} = j$.

Codensity Parity Games. Much as codensity bisimilarities (greatest fixed points of the univariate predicate transformer) are characterized by codensity safety games, codensity equational systems (nested alternating fixed points of the multivariate predicate transformer) are characterized by codensity parity games.

Definition 2.3.2 (Codensity parity game)**.** *The* codensity parity game *of E is the parity game given by Table 3.*

Theorem 2.3.3. *(i, u) is winning for D in the codensity parity game of E if and only if $u \sqsubseteq l_i^{\mathrm{sol}}$.*

The proof of this result involves two intermediate notions that bridge the gap between codensity parity games and progress measures for codensity equational systems : progress measure generators and codensity parity games with ordinals.

Position	Player	Possible moves	Priority
$i \in [1, m]$, $u \in \mathbb{C}_X$	S	i, u, $\sigma \in \Sigma$, $f : X \to \Omega^m_{i,\sigma}$ such that $u \not\sqsubseteq (\tau_{i,\sigma} \circ F_i f \circ c_i)^* \mathbf{\Omega}_{i,\sigma}$	$2i + \delta_{\eta_i = \mu}$
$i \in [1, m]$, $u \in \mathbb{C}_X$ $\sigma \in \Sigma$, $f : X \to \Omega^m_{i,\sigma}$	D	$j \in [1, m]$, $v \in \mathbb{C}_X$ such that $v \not\sqsubseteq f^*_j \mathbf{\Omega}_{i,\sigma}$	0

Table 3: Codensity parity game

Definition 2.3.4 (Progress measure generator)**.** *Assume the setting of Equational System* (1)*. A* progress measure generator *is a set P of tuples $(i, u, (\alpha_1, \ldots, \alpha_k))$ with $i \in [1, m]$, $u \in L_i$ and $(\alpha_1, \ldots, \alpha_k)$ a prioritized ordinal such that*

$$\overline{\alpha}_a = \sup_{(i,u,(\alpha_1,\ldots,\alpha_k)) \in P} \alpha_a$$

$$p_i(\alpha_1, \ldots, \alpha_k) = \bigsqcup_{\substack{(i,u,(\beta_1,\ldots,\beta_k)) \in P \\ (\beta_1,\ldots,\beta_k) \preceq_i (\alpha_1,\ldots,\alpha_k)}} u$$

defines a progress measure $p = ((\overline{\alpha}_1, \ldots, \overline{\alpha}_k), (p_i(\alpha_1, \ldots, \alpha_k)))$.

Let γ be an infinite successor ordinal with cardinality bigger than the supremum of the lengths of the strictly ascending chains in $\mathbb{C}_{X_1}, \ldots, \mathbb{C}_{X_m}$. It can be assumed that the ordinals involved in a progress measure generator for E are all bounded by γ so that, since γ is a successor ordinal, $\overline{\alpha}_k \in \gamma$ for all k. Therefore, the following only deals (implicitly) with ordinals that are bounded by γ.

Note that the actual choice of γ is only a matter of reducing the state space of the codensity parity game with ordinals of E (Definition 2.3.5). Since the prioritized ordinals in this game can actually be ignored (Proposition 2.3.8), γ is of no actual importance.

Definition 2.3.5 (Codensity parity game with ordinals)**.** *The* codensity parity game with ordinals *of E is the safety game given in Table 4.*

This game is called a parity game instead of a safety game because, as shown in Proposition 2.3.8, it is equivalent to a parity game.

To show Theorem 2.3.3, it then suffices to proceed step by step and prove that the following statements are equivalent. It is in this order that the definitions were developed during the internship.

(i) $u \sqsubseteq l^{\mathrm{sol}}_i$;

(ii) there is a prioritized ordinal $(\alpha_1, \ldots, \alpha_k)$ such that $(i, u, (\alpha_1, \ldots, \alpha_k))$ is an element of a progress measure generator of E ;

(iii) there is a prioritized ordinal $(\alpha_1, \ldots, \alpha_k)$ such that $(i, u, (\alpha_1, \ldots, \alpha_k))$ is winning for D in the codensity parity game with ordinals of E ;

(iv) (i, u) is winning for D in the codensity parity game of E.

(i) $\Leftrightarrow$ *(ii)* is a direct consequence of Theorem 1.3.6, and *(iii)* $\Rightarrow$ *(iv)* can be seen easily by well-foundedness of ordinals. Most of the work thus resides in *(ii)* $\Leftrightarrow$ *(iii)* (the generalization of Theorem 2.1.6) and in *(iv)* $\Rightarrow$ *(iii)*.

More particularly, *(ii)* $\Leftrightarrow$ *(iii)* follows naturally from the definitions (the original version of Definition 1.3.5 was designed with this very proof in mind).

Lemma 2.3.6 (*(ii)* $\Rightarrow$ *(iii)*)**.** *The progress measure generators of E are invariants for its codensity parity game with ordinals.*

Lemma 2.3.7 (*(iii)* $\Rightarrow$ *(ii)*)**.** *An invariant for the codensity parity game with ordinals of E is a progress measure generator.*

Finally, for *(iv)* $\Rightarrow$ *(iii)*, the idea is to generalize the proof in the finite case, where the length of a loop in the game (and in particular of a loop with odd maximum priority) can be bounded by the total number of positions in the game. This actually works for any parity game, not just codensity parity games.

Position	Player	Possible moves
$i \in [1,m]$, $u \in \mathbb{C}_X$ $(\alpha_1,\dots,\alpha_k) \in \gamma^k$	S	i, u, $(\alpha_1,\dots,\alpha_k)$, $\sigma \in \Sigma$, $f : X \to \Omega^m_{i,\sigma}$ such that $u \not\sqsubseteq (\tau_{i,\sigma} \circ F_i f \circ c_i)^* \mathbf{\Omega}_{i,\sigma}$
$i \in [1,m] \mid \eta_i = \mu$ $u \in \mathbb{C}_X$ $(\alpha_1,\dots,\alpha_k) \in \gamma^k$ $\sigma \in \Sigma$, $f : X \to \Omega^m_{i,\sigma}$	D	$j \in [1,m]$, $v \in \mathbb{C}_X$, $(\delta_1,\dots,\delta_k) \in \gamma^k$ such that $v \not\sqsubseteq f_j^* \mathbf{\Omega}_{i,\sigma}$ and, for some $(\beta_1,\dots,\beta_k)$, $(\delta_1,\dots,\delta_k) \preceq_j (\beta_1,\dots,\beta_k) \prec_i (\alpha_1,\dots,\alpha_k)$
$i \in [1,m] \mid \eta_i = \nu$ $u \in \mathbb{C}_X$ $(\alpha_1,\dots,\alpha_k) \in \gamma^k$ $\sigma \in \Sigma$, $f : X \to \Omega^m_{i,\sigma}$	D	$j \in [1,m]$, $v \in \mathbb{C}_X$, $(\delta_1,\dots,\delta_k) \in \gamma^k$ such that $v \not\sqsubseteq f_j^* \mathbf{\Omega}_{i,\sigma}$ and, for some $(\beta_1,\dots,\beta_k)$, $(\delta_1,\dots,\delta_k) \preceq_j (\beta_1,\dots,\beta_k) \preceq_i (\alpha_1,\dots,\alpha_k)$

Table 4: Codensity parity game with ordinals

Proposition 2.3.8 ((*iii*) $\Leftrightarrow$ (*iv*))**.** *The codensity parity game of E and its codensity parity game with ordinals are equivalent : a position (i,u) is winning for D in the codensity parity game of E if and only if there is a prioritized ordinal $(\alpha_1,\dots,\alpha_k)$ such that $(i,u,(\alpha_1,\dots,\alpha_k))$ is winning for D in the codensity parity game with ordinals of E.*

Trimmed Codensity Parity Games. Once again, the underlying graphs of codensity parity games may be quite large, but one may trim them with the help of generating sets (Definition 2.1.7).

Definition 2.3.9 (Trimmed codensity parity game)**.** *Have $\mathcal{G}$ be a generating set of $\mathbb{C}_X$. The* trimmed codensity parity game *of E has the same winning conditions as the normal codensity parity game of E of Definition 2.3.2, but D can now only make moves in $\mathcal{G}$, as shown in Table 5.*

The characterization theorem still holds as it is easy to see that any progress measure is generated by a progress measure generator P included in $[1,m] \times \mathcal{G} \times \gamma^k$.

Theorem 2.3.10. *$(i,u) \in [1,m] \times \mathcal{G}$ is winning for D in the trimmed codensity parity game of E if and only if $u \sqsubseteq l_i^{\mathrm{sol}}$.*

This version of the game gives the first interesting instance of codensity parity games.

Example 2.3.11 (Codensity parity game for model checking of $\mathbf{C}\mu_{\Gamma,\Lambda}$ formulæ)**.** *Via Proposition 2.2.7, any equational system involving the semantics of $\mathbf{C}\mu_{\Gamma,\Lambda}$ formulæ can be written as a codensity equational system. The corresponding trimmed codensity parity game (with the generating set from Example 2.1.8) characterizes it : an element x in the base set X has truth value at least ω if and only if δ_x^ω is a winning position in the game.*

In the particular case of classic modal logic, the trimmed codensity parity game is reminescent of the classic parity game for model checking of modal formulæ [18], and is also more compact : Spoiler and Duplicator each play once every fixpoint operator instead of once every propositional connective or modality.

Although parity games for any kind of equational system (and thus in particular for model checking of arbitrary coalgebraic modal logic) already exist [19], the games presented here may be smaller (when the fiber $\mathbb{C}_X$ is bigger then $\mathrm{Hom}(X,\Omega)$), and differ in that they are based on double negation, following the philosophy of codensity. For these reasons, there are plans to submit a paper on model checking of coalgebraic modal logic *via* codensity games to the 24th International Conference on Foundations of Software Science and Computation Structures (FoSSaCS'21) or to the 9th Conference on Algebra and Coalgebra in Computer Science (CALCO'21).

Position	Player	Possible moves	Priority
$i \in [1,m]$, $u \in \mathcal{G}$	S	i, u, $\sigma \in \Sigma$, $f : X \to \Omega^m$ such that $u \not\sqsubseteq (\tau_{i,\sigma} \circ F_i f \circ c_i)^* \mathbf{\Omega}_{i,\sigma}$	$2i + \delta_{\eta_i = \mu}$
$i \in [1,m]$, $u \in \mathcal{G}$ $\sigma \in \Sigma$, $f : X \to \Omega^m$	D	$j \in [1,m]$, $v \in \mathcal{G}$ such that $v \not\sqsubseteq f_j^* \mathbf{\Omega}_{i,\sigma}$	0

Table 5: Trimmed codensity parity game

Transfer of Codensity Equational Systems For completeness, Theorem 2.1.11 was also adapted to codensity equational systems.

Theorem 2.3.12 (Transfer of codensity equational systems)**.** *Let $q : \mathbb{D} \to \mathbb{B}$ be another $\mathbf{CLat}_\sqcap$-fibration and let $T : \mathbb{C} \to \mathbb{D}$ be a full fibered functor from p to q preserving both fibered meets and joins. For all $i \in [1,m]$, $(T\mathbf{\Omega}_i = (T\mathbf{\Omega}_{i,\sigma})_{\sigma \in \Sigma}, \tau_i)$ parameterizes an n-variable codensity lifting along q : write $\left(\widetilde{l}_i^{\mathrm{sol}}\right)_{i \in [1,m]}$ for the solution of the following codensity equational system.*

$$
\begin{aligned}
u_1 &=_{\eta_1} \Phi_{c_1}^{T\mathbf{\Omega}_1, \tau_1}(u_1, \ldots, u_m) \\
&\vdots \\
u_m &=_{\eta_m} \Phi_{c_m}^{T\mathbf{\Omega}_m, \tau_m}(u_1, \ldots, u_m)
\end{aligned}
$$

Recall that $\left(l_i^{\mathrm{sol}}\right)_{i \in [1,m]}$ stands for the solutions of E (Equational System (5)). Then, for all $i \in [1,m]$, $T\left(l_i^{\mathrm{sol}}\right) = \widetilde{l}_i^{\mathrm{sol}}$.

3 Codensity Parity Games for (Bi)similarity

Recall that the original motivation for developing codensity parity games and equational systems was the hope to accomodate and generalize complex bisimulations which codensity bisimilarity could not. For instance, can the fair and delayed bisimulations for Büchi automata be instantiated by codensity equational systems ? If so, it would likely be easy to generalize them to quantitative bisimulation notions for Büchi-like transition systems (by changing the parameters of the predicate transformer), for instance bisimulation distances for Markov chains with Büchi-like acceptance conditions.

Although no positive answer to this question was found, the proof methods used while trying to do so are still interesting in their own right and are thus presented in the following.

3.1 Büchi Automata and (Bi)simulations

Recall first the definitions of Büchi automata and of their bisimulation notions [6]. A Büchi automaton differs from a classic non-deterministic automaton in that they only accept infinite words (instead of finite words) for which there is a corresponding run going an infinite amount of times through accepting states.

Definition 3.1.1 (Büchi automaton)**.** *A* Büchi automaton *is a tuple (X, Σ, c) that consists of*

- *a set X of* states *;*
- *a set Σ of* labels *;*
- *a non-deterministic* transition function $c : X \to 2 \times \mathcal{P}(X)^\Sigma$ *; we write $c_\sigma : X \to \mathcal{P}(X)$ for the function defined by $c_\sigma(x) = (\pi' \circ c)(x)(\sigma)$, $X_0 = (\pi \circ c)^{-1}(0)$ and $X_1 = (\pi \circ c)^{-1}(1)$.*

Let $\mathcal{A} = (X, \Sigma, c)$ be a Büchi automaton. Its bisimulation notions can be defined using the classic bisimulation game.

Definition 3.1.2 (bisimulation game)**.** *The bisimulation game of $\mathcal{A}$ is the safety game given by Table 6.*

Definition 3.1.3 (bisimulations for Büchi automata)**.** *The* (classic) bisimulation *for $\mathcal{A}$ is the*

Position	Player	Possible moves
$(x, y) \in X^2$	S	$\sigma \in \Sigma$ and (x', y) with $x' \in c_\sigma(x)$ or (y', x) with $y' \in c_\sigma(y)$
$(x', y, \sigma) \in X^2 \times \Sigma$	D	(x', y') with $y' \in c_\sigma(y)$

Table 6: Bisimulation game

(equivalence) relation R over X such that two states $x, y \in X$ are related by R (we say x and y are bisimilar) if and only if (x, y) is winning in the bisimulation game of $\mathcal{A}$.

Similarly, the direct / delayed / fair bisimulation *for $\mathcal{A}$ is the (equivalence) relation R over X such that two states $x, y \in X$ are related by R (we say that x and y are directly / delayedly / fairly bisimilar) if and only if there is a winning strategy s for D in the bisimulation game of $\mathcal{A}$ such that any (finite or infinite) play starting from (x, y) and going through positions (x_n, y_n) for S by following s verifies that*

- *(direct bisimulation) $x_n \in X_1 \iff y_n \in X_1$ for all n ;*
- *(delayed bisimulation) if $x_n \in X_1$ there is a $n' \geq n$ such that $y_{n'} \in X_1$ and conversely ;*
- *(fair bisimulation) $\{n \mid x_n \in X_1\}$ is infinite if and only if $\{n \mid y_n \in X_1\}$ is too.*

Example 2.1.3 shows that classic bisimulations can be expressed as codensity bisimilarities, giving way to characterizations *via* codensity safety games and to generalizations, for example as bisimulation distances. The same can be done for direct bisimulations (by changing the coalgebra c to account for accepting states) and for classic and direct simulations, defined similarly from the classic simulation game (by choosing $\mathbf{\Omega}$ to be $\Rightarrow$ instead of $\Leftrightarrow$).

On the contrary, fair and delayed bisimulations cannot be expressed as greatest fixed points of the predicate transformer. The hope is then that they can be expressed as solutions to codensity equational systems. The following focuses on fair bisimulation as an example but the same reasoning applies to delayed (bi)simulations.

3.2 Fair Bisimulations as Codensity Equational Systems

To get a codensity equational system for fair bisimulation, one can start from its parity game, make an equational system out of it using Proposition 1.3.3 and try to express this equational system with the predicate transformer.

The following is the equational system that was derived from a slightly modified but equivalent version of the parity game for fair bisimulation exhibited in [6] : some symmetry was added in the underlying graph and the priorities were changed as to separate different classes of states. As in [6], transitions are assumed to not be labelled, but the generalization of Lemma 3.2.1 to labelled automata is straightforward.

Lemma 3.2.1. *Consider Equational System* (6) *(page 17), where $u^{(b)}_{i,j} : X_i \times X_j \to 2$ and $u^{(b)} = \sum_{i,j} u^{(b)}_{i,j}$, and write $\left(r^{(b)}_{i,j}\right)_{b,i,j}$ for its solution. $\sum_{i,j} r^{(0)}_{i,j}$ is the fair bisimulation relation of $\mathcal{A}$.*

Recall now from Example 2.1.3 that if r^{eq} is the equivalence closure of a relation r over X, then for $x, y \in X$,

$$(\forall x' \in c(x), \exists y' \in c(y), r^{eq}(x', y')) \wedge \\ (\forall y' \in c(y), \exists x' \in c(x), r^{eq}(y', x'))$$

holds if and only if $\Phi_c^{\Leftrightarrow,\exists}(r)(x, y) = 1$. As the fair bisimulation is an equivalence relation, it thus makes sense to generalize the fair bisimulation to other complete lattices Ω as follows. In this definition the role of $\mathbf{\Omega}_{i,j,\epsilon}$ and τ_ϵ is to make sure $\mathrm{fbisim}^{(b)}_{i,j}$ is equal to $\perp$ outside of $X_i \times X_j$.

Definition 3.2.2 (fair Ω-bisimulation)**.** *Let $\mathcal{A} = (X, \Sigma, c : X \to 2 \times FX)$ be a transition system with Büchi-like acceptance conditions for a functor F, and let Ω be a complete lattice. Choose $\tau = (\tau_\sigma)_{\sigma \in \Sigma \cup \{\epsilon\}}$ with $\tau_\sigma : 2 \times F\Omega$ for $\sigma \in \Sigma$ and*

$$\tau_\epsilon : \begin{cases} 2 \times F2 & \to & 2 \\ (b, \_) & \mapsto & b \end{cases}$$

and choose $\mathbf{\Omega}_{i,j} = (\mathbf{\Omega}_{i,j,\sigma})_{\sigma \in \Sigma \cup \{\epsilon\}}$ for all $i, j \in 2$ with $\mathbf{\Omega}_{i,j,\sigma} : \Omega^2 \to \Omega$ depending only on $\sigma \in \Sigma$ (we write $\mathbf{\Omega}_\sigma$) and

$$\mathbf{\Omega}_{i,j,\epsilon} : \begin{cases} 2 \times 2 & \to & \Omega \\ (b_0, b_1) & \mapsto & \begin{cases} \top & \text{if } b_0 = i \wedge b_1 = j \\ \bot & \text{otherwise} \end{cases} \end{cases}$$

Equational System (7) *(read from top to bottom then left to right), computed in $\mathbf{ERel}_\Omega \to \mathbf{Set}$, is called the* equational system for fair Ω-bisimulation *of $\mathcal{A}$ with parameter $(\mathbf{\Omega}, \tau)$.*

We write $\left(\mathrm{fbisim}^{(b)}_{i,j}\right)_{b,i,j}$ for its solution, and $\mathrm{fbisim}^{(0)} = \bigsqcup_{i,j} \mathrm{fbisim}^{(0)}_{i,j} : X^2 \to \Omega$ is called the fair Ω-bisimulation *of $\mathcal{A}$ with parameter $(\mathbf{\Omega}, \tau)$.*

If $\mathrm{fbisim}^{(0)}(x, y) = \omega$ we say that x and y are fairly ω-bisimilar.

Example 3.2.3. *The first important question to ask is then whether the fair 2-bisimulation — with $F = \mathcal{P}$, $\Omega = 2$, $\tau_\sigma = \exists$ and $\mathbf{\Omega}_\sigma = \Leftrightarrow$, $\mathcal{A}$ — instanciates the classic fair bisimulation, as expected.*

For ordinary bisimulations and codensity bisimilarities the answer was straightforward as the univariate predicate transformer instantiated the operator whose greatest fixed point was defined to be the bisimilarity relation.

Here the answer is not that straightforward as the predicate transformer appearing in the equational system for fair 2-bisimulation starts by taking the equivalence closure of its argument, but the equational system for (classic) fair bisimulation does not involve any equivalence closure. The two equational systems are thus different and thus one can not conclude yet as to whether their solutions are the same.

3.3 Codensity Parity Games for Bisimilarity

The codensity equational systems exhibited in Section 3.2 are not exact instances of Definition 2.3.1, but look very similar and as such the results of Section 2.3 still hold. They are also instances of a class of equational systems for which more specific results can be shown : as in Definition 3.2.2, let F be a **Set**-endofunctor, X be a set, $c : X \to 2 \times FX$ be a $2 \times F(-)$-coalgebra, and let Ω be a complete lattice. Write $X_0 = (\pi \circ c)^{-1}(0)$ and $X_1 = (\pi \circ c)^{-1}(1)$. Choose $\tau = (\tau_\sigma)_{\sigma \in \Sigma \cup \{\epsilon\}}$ with $\tau_\sigma : 2 \times F\Omega$ for $\sigma \in \Sigma$ and

$$\tau_\epsilon : \begin{cases} 2 \times F2 & \to & 2 \\ (b, \_) & \mapsto & b \end{cases}$$

and choose $\mathbf{\Omega}_{p,q} = (\mathbf{\Omega}_{p,q,\sigma})_{\sigma \in \Sigma \cup \{\epsilon\}}$ for all $p, q \in 2$ with $\mathbf{\Omega}_{p,q,\sigma} : \Omega^2 \to \Omega$ depending only on $\sigma \in \Sigma$ (we write $\mathbf{\Omega}_\sigma$) and

$$\mathbf{\Omega}_{p,q,\epsilon} : \begin{cases} 2 \times 2 & \to & \Omega \\ (b_0, b_1) & \mapsto & \begin{cases} \top & \text{if } b_0 = p \wedge b_1 = q \\ \bot & \text{otherwise} \end{cases} \end{cases}$$

Definition 3.3.1 (codensity equational system for (bi)similarity)**.** *Let $m \in \mathbb{N}$ be an integer. For each $i \in [1, 4m]$, choose η_i a fixed point operator, two integers $p_i, q_i \in 2$ and two other integers $\chi_i, \xi_i \in [0, m-1]$, such that each triple $(p, q, \chi) \in 2^2 \times [0, m-1]$ appears exactly once in the family $(p_i, q_i, \chi_i)_{i \in [1,4m]}$.*

This gives rise to Equational System (8)*, a* codensity equational system for (bi)similarity*, computed in $\mathbf{ERel}_\Omega \to \mathbf{Set}$.*

$$\begin{aligned} u^{(\chi_1)}_{p_1,q_1} &=_{\eta_1} \Phi_c^{\mathbf{\Omega}_{p_1,q_1},\tau}\left(\bigsqcup_{p,q} u^{(\xi_1)}_{p,q}\right) \\ &\vdots \\ u^{(\chi_{4m})}_{p_{4m},q_{4m}} &=_{\eta_{4m}} \Phi_c^{\mathbf{\Omega}_{p_{4m},q_{4m}},\tau}\left(\bigsqcup_{p,q} u^{(\xi_{4m})}_{p,q}\right) \end{aligned} \tag{8}$$

In the following we fix a codensity equational system for (bi)similarity E as in Definition 3.3.1. We

$$
\begin{aligned}
u^{(0)}_{0,0}(x,y) &=_\nu \Big(\forall x' \in c(x), \exists y' \in c(y), u^{(0)}(x',y')\Big) \wedge \Big(\forall y' \in c(y), \exists x' \in c(x), u^{(0)}(x',y')\Big)\\
u^{(1)}_{0,0}(x,y) &=_\nu \Big(\forall x' \in c(x), \exists y' \in c(y), u^{(1)}(x',y')\Big) \wedge \Big(\forall y' \in c(y), \exists x' \in c(x), u^{(1)}(x',y')\Big)\\
u^{(0)}_{1,0}(x,y) &=_\mu \Big(\forall x' \in c(x), \exists y' \in c(y), u^{(0)}(x',y')\Big) \wedge \Big(\forall y' \in c(y), \exists x' \in c(x), u^{(0)}(x',y')\Big)\\
u^{(1)}_{0,1}(x,y) &=_\mu \Big(\forall x' \in c(x), \exists y' \in c(y), u^{(1)}(x',y')\Big) \wedge \Big(\forall y' \in c(y), \exists x' \in c(x), u^{(1)}(x',y')\Big)\\
u^{(0)}_{0,1}(x,y) &=_\nu \Big(\forall x' \in c(x), \exists y' \in c(y), u^{(1)}(x',y')\Big) \wedge \Big(\forall y' \in c(y), \exists x' \in c(x), u^{(1)}(x',y')\Big)\\
u^{(0)}_{1,1}(x,y) &=_\nu \Big(\forall x' \in c(x), \exists y' \in c(y), u^{(0)}(x',y')\Big) \wedge \Big(\forall y' \in c(y), \exists x' \in c(x), u^{(0)}(x',y')\Big)\\
u^{(1)}_{1,0}(x,y) &=_\nu \Big(\forall x' \in c(x), \exists y' \in c(y), u^{(0)}(x',y')\Big) \wedge \Big(\forall y' \in c(y), \exists x' \in c(x), u^{(0)}(x',y')\Big)\\
u^{(1)}_{1,1}(x,y) &=_\nu \Big(\forall x' \in c(x), \exists y' \in c(y), u^{(1)}(x',y')\Big) \wedge \Big(\forall y' \in c(y), \exists x' \in c(x), u^{(1)}(x',y')\Big)
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
u^{(0)}_{0,0} &=_\nu \Phi_c^{\mathbf{\Omega}_{0,0},\tau}\left(\bigsqcup_{i,j} u^{(0)}_{i,j}\right) & u^{(0)}_{0,1} &=_\nu \Phi_c^{\mathbf{\Omega}_{0,1},\tau}\left(\bigsqcup_{i,j} u^{(1)}_{i,j}\right)\\
u^{(1)}_{0,0} &=_\nu \Phi_c^{\mathbf{\Omega}_{0,0},\tau}\left(\bigsqcup_{i,j} u^{(1)}_{i,j}\right) & u^{(0)}_{1,1} &=_\nu \Phi_c^{\mathbf{\Omega}_{1,1},\tau}\left(\bigsqcup_{i,j} u^{(0)}_{i,j}\right)\\
u^{(0)}_{1,0} &=_\mu \Phi_c^{\mathbf{\Omega}_{1,0},\tau}\left(\bigsqcup_{i,j} u^{(0)}_{i,j}\right) & u^{(1)}_{1,0} &=_\nu \Phi_c^{\mathbf{\Omega}_{1,0},\tau}\left(\bigsqcup_{i,j} u^{(0)}_{i,j}\right)\\
u^{(1)}_{0,1} &=_\mu \Phi_c^{\mathbf{\Omega}_{0,1},\tau}\left(\bigsqcup_{i,j} u^{(1)}_{i,j}\right) & u^{(1)}_{1,1} &=_\nu \Phi_c^{\mathbf{\Omega}_{1,1},\tau}\left(\bigsqcup_{i,j} u^{(1)}_{i,j}\right)
\end{aligned} \tag{7}
$$

write $\left(r^{(\chi_i)}_{p_i,q_i}\right)_{i\in[1,4m]}$ for its solution, and $r^{(\chi)} = \bigsqcup_{p,q} r^{(\chi)}_{p,q}$ for $\chi \in [0, m-1]$.

Definition 3.3.2 (codensity parity game for (bi)similarity)**.** *The* codensity parity game for (bi)similarity *of E is the parity game given by Table 7.*

Theorem 3.3.3. (i,x,y,ω) *is winning for D in the codensity parity game for bisimilarity of E if and only if $\omega \sqsubseteq r^{(\chi_i)}_{p_i,q_i}(x,y)$. In particular, if $(x,y) \notin X_{p_i} \times X_{q_i}$, $r^{(\chi_i)}_{p_i,q_i}(x,y) = \bot$.*

This theorem can now be used to show inequalities on the relations resulting from codensity equational systems for bisimilarity by applying game semantics techniques. The idea is to use the fact that under-approximants of the solutions of codensity equational systems correspond to winning positions, and thus winning strategies, in codensity parity games.

As an example, the proof of Lemma 3.3.4 is given in detail. Note in particular that this lemma is not specific to the $\mathbf{ERel}_\Omega \to \mathbf{Set}$ fibrations but actually works in any $\mathbf{CLat}_\sqcap$-fibration.

Lemma 3.3.4. *Let E and F be two codensity equational systems for bisimilarity respectively given by families $(\eta_i, p_i, q_i, \chi_i, \xi_i)_{i\in[1,4m]}$ and*

Position	Player	Possible moves	Priority
$i \in [1,4m]$ $(x,y) \in X_{p_i} \times X_{q_i}$ $\omega \in \Omega$	S	i, (x,y), ω, $\sigma \in \Sigma$, $f : X \to \Omega$ such that $\omega \not\sqsubseteq (\mathbf{\Omega}_\sigma \circ \Delta(\tau_\sigma \circ Ff \circ c))(x,y)$	$2i + \delta_{\eta_i = \mu}$
$i \in [1,4m]$ $(x,y) \in X_{p_i} \times X_{q_i}$ $\omega \in \Omega$ $\sigma \in \Sigma$ $f : X \to \Omega$	D	$j \in [1,4m]$, $(x',y') \in X_{p_j} \times X_{q_j}$, $\omega' \in \Omega$ such that $\chi_j = \xi_i$ and $\omega' \not\sqsubseteq (\mathbf{\Omega}_\sigma \circ \Delta f)(x',y')$	0

Table 7: Codensity parity game for (bi)similarity

$(\eta'_i, p'_i, q'_i, \chi'_i, \xi'_i)_{i\in[1,4n]}$ *(F, Ω, $\mathbf{\Omega}$, τ and c are the same), and let* $\left(r^{(\chi_i)}_{p_i,q_i}\right)_{i\in[1,4m]}$ *and* $\left(s^{(\chi_i)}_{p_i,q_i}\right)_{i\in[1,4n]}$ *be their respective solutions.*

For a function $\gamma : [0, m-1] \to [0, n-1]$ and $i \in [1,4m]$, define $\Gamma(i) \in [1,4n]$ to be such that $(p_{\Gamma(i)}, q'_{\Gamma(i)}, \chi'_{\Gamma(i)}) = (p_i, q_i, \gamma(\chi_i))$ (it is uniquely defined by the assumption on $(p_j, q_j, \chi_j)_{j\in[1,4n]}$ given in Definition 3.3.1). Assume that

- *for all $i \in [1,4m]$, if $\eta_i = \nu$ then $\eta'_{\Gamma(i)} = \nu$ (and therefore if $\eta'_{\Gamma(i)} = \mu$ then $\eta_i = \mu$) ;*
- *for all $i \in [1,4m]$, $\xi'_{\Gamma(i)} = \gamma(\xi_i)$;*
- *for $i, j \in [1,4m]$, if $\eta_i = \mu$, $\eta_j = \nu$ and $i < j$ then $\Gamma(i) < \Gamma(j)$ (and by the first condition $\eta_{\Gamma(j)} = \nu$).*

Then, for all $p, q \in 2$, $\chi \in [0, m-1]$, $r^{(\chi)}_{p,q} \sqsubseteq s^{(\gamma(\chi))}_{p,q}$.

Proof. By Theorem 3.3.3, we only need to prove that if (i,x,y,ω) is winning for D in the codensity parity game for bisimilarity of E, then $(\Gamma(i),x,y,\omega)$ is also winning for D in the codensity parity game for bisimilarity of F. This last position is well defined as $p_i = p'_{\Gamma(i)}$ and $q_i = q'_{\Gamma(i)}$, by definition of Γ.

Consider a winning strategy for D from (i,x,y,ω) in the game of E. It is also a valid strategy for D from $(\Gamma(i),x,y,\omega)$ in the game of F. Indeed, if S can play (σ, f) from $(\Gamma(i),x,y,\omega)$ in this game, then

$$\omega \not\sqsubseteq (\mathbf{\Omega}_\sigma \circ \Delta(\tau_\sigma \circ Ff \circ c))(x,y)$$

and thus S can also play (σ, f) from (i,x,y,ω) in the game of E. Similarly, if D can answer (j,x',y',ω') in the game of E, then

$$\omega' \not\sqsubseteq (\mathbf{\Omega}_\sigma \circ \Delta f)(x',y')$$

and thus D can answer $(\Gamma(j),x',y',\omega')$ in the game of F, as $\chi'_{\Gamma(j)} = \gamma(\chi_j) = \gamma(\xi_i) = \xi'_{\Gamma(i)}$.

Playing with this strategy, D can in particular never get stuck in the game of F, as it never gets stuck in the game of E. Assume now that there is a play following this strategy and starting from $(\Gamma(i),x,y,\omega)$ in the game of F, going through positions $(\Gamma(i_k),x_k,y_k,\omega_k)_{k\in\mathbb{N}}$ for S, such that for some $i' \in [1,4n]$, $\Gamma(i_k) = i'$ for infinitely many k, and such that $\eta_{i'} = \mu$. Then, in the corresponding play in the game of E, going through positions $(i_k,x_k,y_k,\omega_k)_{k\in\mathbb{N}}$ for S, there is a $i \in \Gamma^{-1}(i')$ such that $i_k = i$ for infinitely many k (by the pigeonhole lemma, since $\Gamma^{-1}(i')$ is finite).

Since $i \in \Gamma^{-1}(i')$ and $\eta_{i'} = \mu$, $\eta_i = \mu$. As the play in the game of E is winning for D (since it follows a winning strategy for D), there is a $j \in [i+1,4m]$ such that $\eta_j = \nu$ and such that $i_k = j$ for infinitely many k, therefore $\Gamma(i_k) = \Gamma(j) = j'$ for infinitely many k, and $\eta'_{j'} = \nu$. Finally, since $i < j$, $\eta_i = \mu$ and $\eta_j = \nu$, $i' = \Gamma(i) < \Gamma(j) = j'$.

We thus showed that the considered play in the game of F is winning for D, and therefore that all plays following the strategy and starting from $(\Gamma(i),x,y,\omega)$ in this game are winning for D :

$(\Gamma(i), x, y, \omega)$ is winning for D in the game of F. $\square$

Using similar proof methods one may for example also show that when the Ω-relations $\boldsymbol{\Omega}_\sigma$ are equivalence relations (a notion that can be defined as soon as Ω is a quantale), then so is the fair Ω-bisimulation.

Unfortunately, the same proof method also shows that fair Ω-bisimulations are actually ordinary bisimulations as soon as the $\boldsymbol{\Omega}_\sigma$ are symmetric, and thus that Definition 3.2.2 fails to instantiate the fair bisimulation for Büchi automata. There are other ways to define fair (bi)simulations, but nothing satisfying was found. For example, fair Ω-simulations (defined like in Section 3.2 but starting from the parity game for fair simulation) may not suffer from the same shortcoming, but it is still not known whether they instantiate classic fair simulations.

Some additional time will still be put into trying to instantiate the fair and delayed bisimulations with codensity equational systems, as such a definition would give rise to new notions of quantitative fair and delayed bisimulations for Büchi-like automata, for which the codensity parity game point of view could give intuitive methods of showing soundness results (for instance, showing that if two states are fairly ω-bisimilar then the truth values of a coalgebraic modal formula differ at most by ω).

Conclusion

In fine, we developed the notion of codensity equational systems, a formalism for nested alternating fixed points involving codensity liftings, and showed that their solutions are characterized by the winning positions of well-crafted parity games called codensity parity games. From this result, we derived a new kind of parity games for model checking of coalgebraic modal logic, reminescent of the classic model checking game for classic modal logic, and which will be the topic of a future paper.

Unfortunately, we did not manage to instantiate fair and delayed bisimulations, our main motivation for developing codensity equational systems. This will be the subject of future research.
Still, our work on this question lead to the use of an interesting proof method that uses game semantics to derive inequalities. Formalizing and studying this proof method in a categorical framework adapted to game theory may also be the subject of future research.

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